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Direct Estimates on Intersection Probabilities of Random Walks

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A study is made of intersection properties of independent random walks in d-dimensional lattice space Z^d. A simple method is developed which makes it possible to estimate intersection probalities of two random walks with killing rate m directly. It is expected that the method can be generalized and extended to other issues.

KEY WORDS: Random walks; intersection probabilities; decoupling identity; path splitting method.

1. INTRODUCTION: NOTATION AND MAIN RESULTS

Intersection properties of random walks or Brownian paths have been investigated extensively by many authors[1,7–9,10–13] because, in addition to their own attraction, their properties have been known to be intimately related to quantum field theory and statistical mechanics.[3–5] Previous methods in the study of the intersection properties of random walks have involved probabilistic arguments[7–9,11–13] and renormalization group approaches,[1,10] which are not so elementary.

The aim of this paper is to present a simple method by which upper and lower bounds of intersection probabilities of two random walks with killing rate (mass) m can be estimated directly. Although the results of this paper are known,[10–12] the method is simple and gives the results easily. I now describe briefly the main ideas used in this paper. Let ω and ω' be simple random walks in Z^d. Then either ω and ω' intersect each other or else they are avoiding. Thus, the decoupling identity

$$1 = \chi(\omega \cap \omega' = \varnothing) + \chi(\omega \cap \omega' \neq \varnothing) \tag{1.1}$$

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holds. Next, assume that $\omega \cap \omega' \neq \varnothing$. We may split the walk ω such that

$$\omega = \omega_1 \cup \omega_2 \tag{1.2}$$

where ω_1 does not intersect ω' except at the endpoint, and ω_2 has no restrictions. Thus the sum over ω_2 will be factorized. The idea of the path (walk) splitting method essentially the same as the above was also employed by Lawler under the name of the modified stopping time.[12] Here I use the decoupling identity and the path splitting method repeatedly and systematically to obtain explicit bounds on intersection probabilities. Detailed descriptions are postponed to the next section.

I believe that the method used in this paper can be generalized and extended to other issues, such as intersection properties of random paths and the diffusion of self-avoiding walks. In fact, I initiated the study of the method to investigate self-avoiding walks.[14] It turns out that an application of the method gives a simple derivation of the Brydges–Spencer lace expansion.[6] It may be possible that Slade's results and its proofs[15] on the diffusion of self-avoiding random walks can be improved by using this method.

I now introduce the notation. A random walk on Z^d is a finite sequence ω: $\{t_1, t_1+1,..., t_2\} \to Z^d$, written as $\omega = \{\omega(t_1), \omega(t_1+1),..., \omega(t_2)\}$, with $|\omega(i+1) - \omega(i)| = 1$, where, for $x, y \in Z^d$, $|x-y|$ is the Euclidean distance between x and y. Let $|\omega|$ be the number of the steps (not the sites) of ω. For given $x, y \in Z^d$ and $t_1, t_2 \in Z^+$ with $t_1 \leqslant t_2$, write

$$W(x, y; t_1, t_2) = \{\omega: \{t_1,..., t_2\} \to Z^d: \omega(t_1) = x, \omega(t_2) = y, |\omega(i+1) - \omega(i)| = 1\} \tag{1.3}$$

and

$$W(x, y) = \bigcup_{t \in Z^+} W(x, y; 0, t) \tag{1.4}$$

$$W(x) = \bigcup_{y \in Z^d} W(x, y)$$

Denote the sets of random walks which do not visit the starting site after the first step by

$$W_1(x, y; t_1, t_2) = \{\omega \in W(x, y; t_1, t_2): \omega(i) \neq x \text{ if } i \neq t_1\}$$

$$W_1(x, y) = \bigcup_{t \in Z^+} W_1(x, y; 0, t) \tag{1.5}$$

$$W_1(x) = \bigcup_{y \in Z^d} W_1(x, y)$$

Next let us define the intersection probability between simple random walks with killing rate $m > 0$.[10] For a given $m > 0$, let $\xi = (2d + m^2)^{-1}$ and

$$\chi(m) = \sum_{\omega \in W(0)} \xi^{|\omega|} \tag{1.6}$$

where the term corresponding to $|\omega| = 0$ equals to 1. The intersection probability for two random walks starting at 0 and x, respectively, is given by

$$P(0, x; m) = \chi(m)^{-2} \sum_{\substack{\omega \in W(0) \\ \omega' \in W(x)}} \xi^{|\omega| + |\omega'|} \chi(\omega \cap \omega' \neq \varnothing) \tag{1.7}$$

and the average intersection probability for two walks is given by

$$g(m) = \sum_x m^d P(0, x; m) \tag{1.8}$$

We then have the following result.[10]

Theorem 1.1. There exist positive constants c_1, c_2, c_3, c_1', c_2', and c_3' independent of m such that the bounds

$$\left.\begin{matrix} c_1 |\ln m|^{-1} \\ c_2 m^{d-4} \\ c_3 \end{matrix}\right\} \leqslant g(m) \leqslant \begin{cases} c_1' |\ln m|^{-1} & d = 4 \\ c_2' m^{d-4}, & d > 4 \\ c_3', & d < 4 \end{cases}$$

hold for small m.

Remark. (a) Similar results to the above have been obtained by Felder and Fröhlich[10] by using a renormalization group equation. See also Aizenman.[1]

(b) For any dimension, explicit upper and lower bounds for $g(m)$ are given in Proposition 3.1 in Section 3.

Now consider the probabilities that two random walks starting at the origin do not meet at a common site after the first step. We write that

$$\chi_1(m) = \sum_{\omega \in W_1(0)} \xi^{|\omega|} \tag{1.9}$$

and

$$P_2(m) = \chi_1(m)^{-2} \sum_{\omega_1, \omega_2 \in W_1(0)} \xi^{|\omega_1| + |\omega_2|} \chi(\omega_1 \cap (\omega_2 - \{0\}) = \varnothing) \tag{1.10}$$

where in the summands the terms corresponding to the walks with no step equal to 1.

We have the following result.

Theorem 1.2. There exist positive constants c_1, c_2, c_3, c_1', c_2', and c_3' independent of m such that the bounds

$$\left.\begin{matrix} c_1 |\ln m|^{-1} \\ c_2 m^{4-d} \\ c_3 \end{matrix}\right\} \leqslant p_2(m) \leqslant \begin{cases} c_1' |\ln m|^{-1/2} & d=4 \\ c_2' m^{(4-d)/2}, & d=3 \\ c_3', & d>4 \end{cases}$$

hold for small m.

Remark. (a) For walks of fixed length, results analogous to those in Theorem 1.2 have been obtained by Lawler[11-113] by using probability arguments.

(b) Explicit upper and lower bounds for $P_2(m)$ are given in Proposition 2.2.

The contents of the paper are as follows: In Section 2, I describe the method in detail, and then prove Theorem 1.2. The proof of Theorem 1.1 is given in Section 3. Finally, I give a brief description on possible generalizations and applications of the method in Section 4.

2. INTERSECTIONS OF TWO RANDOM WALKS

In this section I describe the main ideas used in this paper in detail, and then obtain explicit upper and lower bounds for $P_2(m)$, from which Theorem 1.2 will follow as a corollary. I first introduce more notation. Let us denote the set of random loops by $L(x)$,

$$L(x) = \bigcup_{t=0}^{\infty} W(x, x; 0, t) \tag{2.1}$$

and let

$$D(k) = \frac{1}{d} \sum_{j=1}^{d} \cos(k_j) \tag{2.2}$$

Recall the definition of $W(x, y)$ and $W_1(x, y)$ in (1.4) and (1.5), respectively. For a given $\xi = (2d + m^2)^{-1}$, let

$$\begin{aligned} G(x, y; m) &= \sum_{\omega \in W(x, y)} \xi^{|\omega|} \\ G_1(x, y; m) &= \sum_{\omega \in W_1(x, y)} \xi^{|\omega|} \end{aligned} \tag{2.3}$$

Then it is well known that

$$G(x, y; m) = \int_{[-\pi, \pi]^d} dk\, [1 - 2d\xi D(k)]^{-1} e^{ik \cdot (y-x)} \tag{2.4}$$

We write that

$$\begin{aligned} l(m) &= \sum_{\omega \in L(x)} \xi^{|\omega|} \\ \Gamma_2(m) &= \sum_{y \in Z^d} G(x, y; m)^2 \\ \tilde{\Gamma}_2(m) &= \sum_{y \in Z^d} G_1(x, y; m)^2 \end{aligned} \tag{2.5}$$

Then it follows from an inspection that

$$G(x, y; m) = l(m)^2\, G_1(x, y; m) \tag{2.6}$$

For a given random walk $\omega \in W(x, y; 0, t)$, one defines $r\omega$ to be the random walk with reverse ordering:

$$r\omega = \{(r\omega)(0) = \omega(t), (r\omega)(1) = \omega(t-1), \ldots, (r\omega)(t) = \omega(0)\}$$

Thus, if $\omega \in W(x, y; 0, t)$, then $r\omega \in W(y, x; 0, t)$. For any $\omega \in W(x, y; 0, t)$, let $\omega - \omega(0)$ be the random walk obtained by restricting ω on $\{1, \ldots, t\}$: $\omega - \omega(0) = \{\omega(1), \omega(2), \ldots, \omega(t)\}$. The quantity $\omega - \omega(t)$ is defined analogously.

I now turn to the description of the main idea in this paper. Consider the quantities defined by

$$H(m) \equiv \sum_{\omega, \omega' \in W_1(0)} \xi^{|\omega| + |\omega'|} \chi(\omega' \cap (\omega - \omega(0)) = \varnothing) \tag{2.7}$$

$$\tilde{H}(m) \equiv \sum_{\omega, \omega' \in W_1(0)} \xi^{|\omega| + |\omega'|} \chi(\omega' \cap (\omega - \omega(0)) \neq \varnothing) \tag{2.8}$$

From the definition of $P_2(m)$ in (1.10), it follows that

$$P_2(m) = \chi_1(m)^{-2}\, H(m) \tag{2.9}$$

Using the decoupling identity

$$1 = \chi(\omega' \cap (\omega - \omega(0)) = \varnothing) + \chi(\omega' \cap (\omega - \omega(0)) \neq \varnothing) \tag{2.10}$$

we obtain that

$$\chi_1(m)^2 = H(m) + \tilde{H}(m) \tag{2.11}$$

For given $\omega \in W_1(0, y; 0, t)$ and $\omega' \in W_1(0. y'; 0, t')$ with $\omega' \cap (\omega - \omega(0)) \neq \emptyset$, let s be the *latest step* (time) at which $\omega - \omega(0)$ intersects ω'. Let $x = \omega(s)$ and $s' = \max\{t'': \omega'(t'') = x\}$. Let us split the random walks ω and ω' such that

$$\omega = \omega_1 \cup \omega_2, \qquad \omega' = \omega_1' \cup \omega_2' \tag{2.12}$$

where $\omega_1 \in W_1(0, x; 0, s)$, $\omega_2 \in W_1(x, y; s, t)$, $\omega_1' \in W_1(0, x; 0, s')$, and $\omega_2' \in W_1(x, y'; s', t')$ with $\omega_1' \cap (\omega_2 - \omega_2(s)) = \emptyset$, $\omega_2' \cap (\omega_2 - \omega_2(s)) = \emptyset$, and $0 \notin \omega_2 \cup \omega_2'$. Using (2.12), the reverse ordering invariance, and summing over ω_1 and x, we obtain from (2.8) that

$$\begin{aligned}\tilde{H}(m) = \sum_{x \in Z^d} G_1(0, x) \sum_{\substack{\omega_1' \in W(x,0): \\ r\omega_1' \in W_1(0, x)}} \sum_{\substack{\omega_2 \omega_2' \in \omega_1'(x): \\ 0 \notin \omega_2 \cup \omega_2'}} \xi^{|\omega_1'| + |\omega_2| + |\omega_2'|} \\ \times \chi(\omega_1' \cap (\omega_2 - \omega_2(0)) = \emptyset)\, \chi(\omega_2' \cap (\omega_2 - \omega_2(0)) = \emptyset)\end{aligned} \tag{2.13}$$

It may be instructive to represent $\tilde{H}(m)$ diagrammatically (see Fig. 1a). Notice that the sum over ω_1 has been factorized to given $G_1(0, x)$. We write

$$\tilde{H}(m) = \tilde{H}'(m) - \tilde{H}_1(m) \tag{2.14}$$

where $\tilde{H}'(m)$ is defined by removing the restriction $0 \notin \omega_2 \cup \omega_2'$ from the definition of $\tilde{H}(m)$ in (2.13), and $\tilde{H}_1(m)$ is defined by replacing $0 \notin \omega_2 \cup \omega_2'$ by $0 \in \omega_2 \cup \omega_2'$ in the definition of $\tilde{H}(m)$.

Let us decompose $\tilde{H}'(m)$ further. We use the decoupling identity

$$\chi(\omega_1' \cap (\omega_2 - \omega_2(0)) = \emptyset) = 1 - \chi(\omega_1' \cap (\omega_2 - \omega_2(0)) \neq \emptyset) \tag{2.15}$$

and the splitting method similar to that in (2.12): For given $\omega_1' \in W(x)$ and $\omega_2 \in W_1(x)$ with $\omega_1' \cap (\omega_2 - \omega_2(0)) \neq \emptyset$, let $s' > 0$ be the *eariest step* (time) at which ω_1' intersects $\omega_2 - \omega_2(0)$ at $y = \omega_1'(s')$. Then one may split the walk ω_1' and $\omega_1' = \omega_3 \cup \omega_4$, where $\omega_3 \in W(x, y; 0, s')$ with $r\omega \in W_1(y, x; 0, s')$, $\omega_4 \in W(y)$, and $(\omega_3 - \omega_3(s') \cap (\omega_2 - \omega_2(0)) = \emptyset$. Substituting (2.15) into $\tilde{H}'(m)$ and using the above splitting method, we obtain from (2.13) that

$$\tilde{H}'(m) = \tilde{H}_2(m) - \tilde{H}_3(m) \tag{2.16}$$

where

$$\tilde{H}_2(m) = \sum_{\substack{x: \\ x \neq 0}} G_1(0, x)^2 \sum_{\omega_2, \omega_2' \in W_1(x)} \xi^{|\omega_2'| + |\omega_2'|} \chi(\omega_2' \cap (\omega_2 - \omega_2(0)) = \emptyset) \tag{2.17}$$

and

$$\tilde{H}_3(m) = \sum_{\substack{x: \\ x \neq 0}} G_1(0, x) \sum_{\omega_2, \omega_2' \in W_1(x)} \sum_{y \in \omega_2 - \omega_2(0)} G_1(0, y)$$
$$\times \sum_{\substack{\omega_3 \in W(x, y): \\ r\omega_3 \in W_1(y, x),\, 0 \notin \omega_3}} \xi^{|\omega_2| + |\omega_2'| + |\omega_3|} \chi((\omega_2' \cap (\omega_2 - \omega_2(0)) = \varnothing)$$
$$\times \chi((\omega_3 - \{y\}) \cap (\omega_2 - \omega_2(0)) = \varnothing\} \tag{2.18}$$

The quantities $\tilde{H}$ and $\tilde{H}_3$ may be represented diagrammatically as in Fig. 1.
From (2.17) it follows that

$$\tilde{H}_2(m) = \left[\sum_{x \neq 0} G_1(0, x)^2 \right] H(m) \tag{2.19}$$

and so an inspection shows that

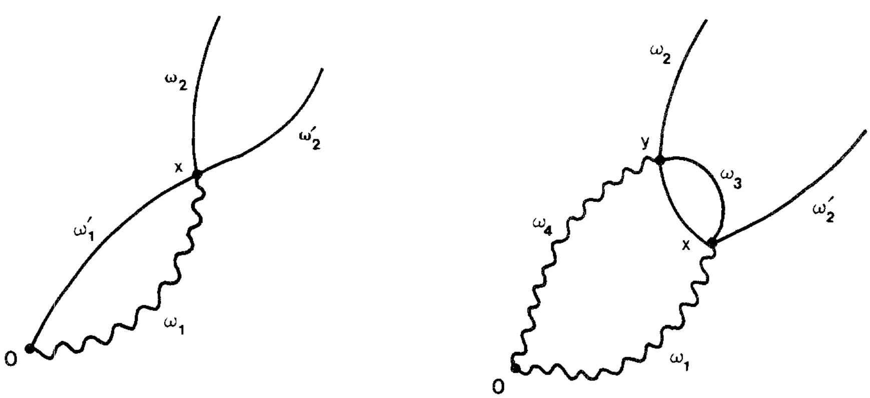
$$H(m) + \tilde{H}_2(m) = \tilde{\Gamma}_2(m)\, H(m) \tag{2.20}$$

The above result is summarized as follows.

Lemma 2.1. Let $H(m)$, $\tilde{H}_1(m)$, and $\tilde{H}_3(m)$ be defined as in (2.7), (2.14), and (2.18), respectively. Then the relation

$$\chi_1(m)^2 = \tilde{\Gamma}_2(m)\, H(m) - \tilde{H}_1(m) - \tilde{H}_3(m)$$

holds



a) $\tilde{H}(m)$ b) $\tilde{H}_3(m)$

Figure 1

Proof. The lemma follows from (2.11), (2.14), (2.16), and (2.20).

Let us turn to the proof of Theorem 1.2. The main result in this section are the following explicit bounds for $P_2(m)$.

Proposition 2.2. Let $P_2(m)$ be defined as in (1.10). Then the bounds

$$\tilde{\Gamma}_2(m)^{-1} \leqslant P_2(m) \leqslant l(m)^{3/2}/\tilde{\Gamma}_2(m)^{1/2}$$

hold.

Proof of Theorem 1.2. From (2.5) and (2.6) it follows that

$$\begin{aligned} \Gamma_2(m) &= l(m)^2\, \tilde{\Gamma}_2(m) \\ \chi(m) &= l(m)\, \chi_1(m) \end{aligned} \tag{2.21}$$

Also, it is well known that

$$\frac{1}{\pi^2 d} k^2 \leqslant 1 - D(k) \leqslant \frac{1}{2d} k^2 \tag{2.22}$$

on $[-\pi, \pi]^d$. The definitions of $\Gamma_2(m)$ and $l(m)$ in (2.5) and (2.6) imply that

$$\begin{aligned} \Gamma_2(m) &= (2\pi)^{-d} \int_{[-\pi,\pi]} dk \left[1 - \frac{2d}{2d+m^2} D(K)\right]^{-2} \\ l(m) &= (2\pi)^{-d} \int_{[-\pi,\pi]^d} dk \left[1 - \frac{2d}{2d+m^2} D(k)\right]^{-1} \\ \chi(m) &= (2d+m^2)/m^2 \end{aligned} \tag{2.23}$$

The proof of Theorem 1.2 follows from (2.21)–(2.23) and Proposition 2.2.

The rest of this section is devoted to a proof of Proposition 2.2.

Proof of Proposition 2.2. (a) *Lower bound.* The lower bound follows from Lemma 2.2 and the fact that $\tilde{H}_1(m) + \tilde{H}_3(m) \geqslant 0$.

(b) *Upper bound:* In order to obtain the upper bound, we use Lemma 2.1 and the Schwarz inequality. The method of the Schwarz inequality has also been used by Lawler[11,12] for walks of a fixed length. Recall the definition of $H(m)$ in (2.7). Using the Schwarz inequality and the fact that $w_1(0) \subset w(0)$, we obtain the bound

$$H(m)^2 \leqslant \chi_1(m)\, Q(m) \tag{2.24}$$

where

$$Q(m) = \sum_{\omega_2 \in W_1(0)} \sum_{\substack{\omega'_1 \in W(0) \\ \omega'_2 \in W_1(0)}} \xi^{|\omega_2| + |\omega'_1| + |\omega'_2|} \times \chi(\omega'_1 \cap (\omega_2 - \omega_2(0)) = \varnothing)\, \chi(\omega'_2 \cap (\omega_2 - \omega_2(0)) = \varnothing) \quad (2.25)$$

We then use the decoupling identity (2.15), the splitting method $\omega'_1 = \omega_3 \cap \omega_4$ as below (2.15), and sum over ω_4 to conclude that

$$Q(m) = \chi(m)\, H(m) - \chi(m)\, Q_2(m) \quad (2.26)$$

where

$$Q_2(m) = \sum_{\omega_2, \omega'_2 \in W_1(0)} \sum_{y} \sum_{\substack{\omega_3 \in W(0, y): \\ r\omega_3 \in W(y, 0)}} \xi^{|\omega_2| + |\omega'_2| + |\omega_3|} \times \chi(\omega'_2 \cap (\omega_2 - \omega_2(0)) = \varnothing)\, \chi((\omega_3 - \{y\}) \cap (\omega_2 - \omega_2(0)) = \varnothing) \quad (2.27)$$

Combining (2.24), (2.26), and Lemma 2.1, we obtain

$$\begin{aligned} H(m)^2 &\leqslant \chi_1(m)\, \chi(m)[H(m) - Q_2(m)] \\ &= \chi_1(m)\, \chi(m)\{[\chi_1(m)^2 + \tilde{H}_1(m)]/\tilde{\Gamma}_2(m)\} \\ &\quad \times \chi_1(m)\, \chi(m)\{[\tilde{H}_3(m)/\tilde{\Gamma}_2(m)] - Q_2(m)\} \end{aligned} \quad (2.28)$$

Using the translational invariance and the fact that $\sum_x G_1(0, x)\, G_1(x, y - x) \leqslant \tilde{\Gamma}_2(m)$, we obtain from (2.18) and (2.27) that

$$\tilde{H}_3(m) \leqslant \tilde{\Gamma}_2(m)\, Q_2(m) \quad (2.29)$$

Next, consider $\tilde{H}_1(m)$, which has been defined by replacing $0 \notin \omega_2 \cup \omega'_2$ by $0 \in \omega_2 \cup \omega'_2$ in the definition of $\tilde{H}(m)$ in (2.13). By undoing the splitting, we obtain that

$$\tilde{H}_1(m) \leqslant \sum_{\substack{\omega, \omega' \in W(0) \\ 0 \in (\omega - \omega(0)) \cup (\omega' - \omega'(0))}} \xi^{|\omega| + |\omega'|}$$

and so

$$\chi_1(m)^2 + \tilde{H}_1(m) \leqslant \chi(m)^2 \quad (2.30)$$

Substituting (2.29) and (2.30) into (2.28), and using (2.9) and the fact that $\chi(m) = l(m)\, \chi_1(m)$, we obtain the upper bound in the proposition. This completes the proof.

3. AVERAGE INTERSECTION PROBABILITY OF TWO RANDOM WALKS

We obtain explicit upper and lower bounds for $g(m)$ in this section. Then Theorem 1.1 will follow as a corollary. A upper bound of $g(m)$ is obtained by employing a method similar to that used to obtain the equality in (2.26). In order to get a lower bound, we use the method suggested by Aizenman (see the proof of Proposition 7.2 of ref. 1). The following is the main result in this section.

Proposition 3.1. Let $g(m)$ be defined as in (1.8). Then the bounds

$$\tfrac{1}{4}m^d\chi_1(m)^2/\tilde{\Gamma}_2(m) \leqslant g(m) \leqslant m^d\chi(m)^2/\tilde{\Gamma}_2(m)$$

hold.

Proof of Theorem 1.1. The theorem follows from (2.21)–(2.23) and the above proposition.

I now produce the proof of Proposition 3.1.

Proof of Proposition 3.1. (a) *Upper bound:* Recall the definition of $g(m)$ in (1.8). We use the splitting method similar to that used in (2.12): For given $\omega \in W(0, z; 0, t)$ and $\omega' \in W(x, z'; 0, t)$ with $\omega \cap \omega' \neq \varnothing$, let s be the latest step at which ω intersects ω' at $y = \omega(s)$, and let $s' = \max\{t'': \omega'(t'') = y\}$. We split ω and ω' as

$$\omega = \omega_1 \cup \omega_2 \qquad \text{and} \qquad \omega' = \omega'_1 \cup \omega'_2$$

where $\omega_1 \in W(0, y; 0, s)$, $\omega_2 \in W_1(y, z; s, t)$, $\omega'_1 \in W(x, y; 0, s')$, and $\omega'_2 \in W_1(y, z'; s', t')$ with $\omega'_1 \cap (\omega_2 - \omega_2(s)) = \varnothing$ and $\omega'_2 \cap (\omega_2 - \omega_2(s)) = \varnothing$. Using the above splittings, translational and reverse ordering invariance, and summing over ω_1, it follows from (1.7) and (1.8) that

$$g(m) = m^d\chi(m)^{-1}\, Q(m)$$

where $Q(m)$ has been defined in (2.25). We now use (2.26), Lemma 2.1, (2.29), and (2.30) to conclude that

$$\begin{aligned} g(m) &= m^d[H(m) - Q_2(m)] \\ &\leqslant m^d\{\chi_1(m)^2 + \tilde{H}_1(m)/\tilde{\Gamma}_2(m)\} + m^d\{[\tilde{H}_3(m)/\tilde{\Gamma}_2(m)] - Q_2(m)\} \\ &\leqslant m^d\chi(m)^2/\tilde{\Gamma}_2(m) \end{aligned}$$

This give the upper bound in the proposition.

(b) *Lower bound:* We use a method similar to that used in the proof of Proposition 7.2 of ref. 1. Let $V(\omega, \omega')$ be the cardinality of $\{z: z \in \omega \cap \omega'\}$. Then $\chi(V(\omega, \omega') \neq 0) = \chi(\omega \cap \omega' \neq \varnothing)$. Using the Schwarz inequality, we obtain the following bound for $g(m)$:

$$g(m) = m^d \chi(m)^{-2} \sum_x \sum_{\substack{\omega \in W(0) \\ \omega' \in W(x)}} \xi^{|\omega| + |\omega'|} \chi(V(\omega, \omega') \neq 0)$$

$$\geqslant m^d \chi(m)^{-2} A(m)/B(m) \tag{3.1}$$

where

$$A(m) = \left[\sum_x \sum_{\substack{\omega \in W(0) \\ \omega' \in W(x)}} \xi^{|\omega| + |\omega'|} V(\omega, \omega') \right]^2$$

$$B(m) = \sum_x \sum_{\substack{\omega \in W(0) \\ \omega' \in W(x)}} \xi^{|\omega| + |\omega'|} V(\omega, \omega')^2 \tag{3.2}$$

To estimate $B(m)$, we write

$$B(m) = \sum_{z_1, z_2} \sum_x \sum_{\substack{\omega \in W(0) \\ \omega \in W(x)}} \xi^{|\omega| + |\omega'|} \chi(z_1, z_2 \in \omega \cap \omega')$$

We split the paths conditioning on the site of the first hit. It is easily obtained that

$$B(m) = 2\chi(m)^2 \sum_{z_1, z_2} \sum_x G_1(0, z_1)\, G_1(z_1, z_2)$$

$$\times \{G_1(x, z_1)\, G_1(z_1, z_2) + G_1(x, z_2)\, G_1(z_2, z_1)\}$$

$$= 4\chi(m)^2\, \chi_1(m)^2\, \Gamma_1(m) \tag{3.3}$$

Here we have suppressed m in the notation $G(x, y)$. Similarly, we estimate $A(m)$ by

$$A(m) = \left(\sum_z \sum_x \sum_{\substack{\omega \in W(0) \\ \omega' \in W(x)}} \xi^{|\omega| + |\omega'|} \chi(z \in \omega \cap \omega') \right)^2$$

$$= \left[\sum_z \sum_x G_1(0, z)\, G_1(x, z) \right]^2 \chi(m)^4$$

$$= \chi_1(m)^4\, \chi(m)^4 \tag{3.4}$$

From (3.1), (3.3), and (3.4), it follows that

$$g(m) \geqslant \tfrac{1}{4} m^d \chi_1(m)^2 / \tilde{\Gamma}_2(m) \tag{3.5}$$

The lower bound follows from (2.21) and the above inequality. This completes the proof of the proposition.

4. DISCUSSION

I have derived the upper and lower bounds on various intersection probabilities of random walks by using the decoupling identity and the path splitting method. As mentioned in the Introduction, the Brydges–Spencer lace expansion[6] for self-avoiding random walks can be derived easily by using the method employed in this paper.[14] Possibly Slade's result[15] can be improved by a refinement of the method.

It would be interesting to see an extension of the method to random paths. For random walks I have used countable additivity implicitly. Since random paths have continuous time parameters, a direct extension of the method to random paths seems to be very difficult.

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